

Darboux transformations and global explicit solutions for nonlocal Davey-Stewartson I equation

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Abstract

For the nonlocal Davey-Stewartson I equation, the Darboux transformation is considered and explicit expressions of the solutions are obtained. Like the nonlocal equations in 1+1 dimensions, many solutions may have singularities. However, by suitable choice of parameters in the solutions of the Lax pair, it is proved that the solutions obtained from seed solutions which are zero and an exponential function of t respectively, by a Darboux transformation of degree n are global solutions of the nonlocal Davey-Stewartson I equation. The derived solutions are soliton solutions when the seed solution is zero, in the sense that they are bounded and have n peaks, and “line dark soliton” solutions when the seed solution is an exponential function of t , in the sense that they are bounded and their norms change fast along some straight lines.

1 Introduction

In [1], Ablowitz and Musslimani introduced the nonlocal nonlinear Schrödinger equation and got its explicit solutions by inverse scattering. Quite a lot of work were done after that for this equation and other equations.[2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

In [12], Fokas studied high dimensional equations and introduced the nonlocal Davey-Stewartson I equation

$$\begin{aligned} iu_t &= u_{xx} + u_{yy} + 2\sigma u^2 \bar{u}^* + 2uw_y, \\ w_{xx} - w_{yy} &= 2\sigma(u\bar{u}^*)_y, \end{aligned} \tag{1}$$

where w satisfies $\bar{w}^* = w$. Here $\bar{f}(x, y, t) = f(-x, y, t)$ for a function f , $*$ refers to complex conjugation. The solution of (1) is PT symmetric in the sense that if $(u(x, y, t), w(x, y, t))$

is a solution of (1), then so is $(u^*(-x, y, -t), w^*(-x, y, -t))$. This leads to a conserved density $u\bar{u}^*$, which is invariant under $x \rightarrow -x$ together with complex conjugation.

As is known, the usual Davey-Stewartson I equation does not possess a Darboux transformation in differential form. Instead, it has a binary Darboux transformation in integral form.[13, 14] However, for the nonlocal Davey-Stewartson I equation (1), we can construct a Darboux transformation in differential form. Like the nonlocal equations in 1+1 dimensions, the solutions may have singularities. Starting from the seed solutions which are zero and an exponential function of t , we prove that the derived solutions can be globally defined and bounded for all $(x, y, t) \in \mathbf{R}^3$ if the parameters are suitably chosen. Unlike the usual Davey-Stewartson I equation where localized solutions are dromion solutions if the seed solution is zero,[15, 16, 17] the derived solutions here are soliton solutions in the sense that there are n peaks in the solutions obtained from a Darboux transformation of degree n . If the seed solution is an exponential function of t , the norms of the derived solutions change a lot along some straight lines. We call them “line dark soliton” solutions.

In Section 2 of this paper, the Lax pair for the nonlocal Davey-Stewartson I equation is reviewed and its symmetries are considered. Then the Darboux transformation is constructed and the explicit expressions of the new solutions are derived. In Section 3 and Section 4, the soliton solutions and “line dark soliton” solutions are constructed respectively. The globalness, boundedness and the asymptotic behaviors of those solutions are discussed.

2 Lax pair and Darboux transformation

Consider the 2×2 linear system

$$\begin{aligned}\Phi_x &= \tau J \Phi_y + \tau P \Phi = \tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_y + \tau \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} \Phi, \\ \Phi_t &= -2i\tau^2 J \Phi_{yy} - 2i\tau^2 P \Phi_y + iQ \Phi \\ &= -2i\tau^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{yy} - 2i\tau^2 \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} \Phi_y \\ &\quad + i\tau \begin{pmatrix} -\tau uv - \tau w_y - w_x & u_x + \tau u_y \\ v_x - \tau v_y & \tau uv + \tau w_y - w_x \end{pmatrix} \Phi\end{aligned}\tag{2}$$

where $\tau = \pm 1$, u, v, w are functions of (x, y, t) . The compatibility condition $\Phi_{xt} = \Phi_{tx}$ gives the evolution equation

$$\begin{aligned}iu_t &= u_{xx} + \tau^2 u_{yy} + 2\tau^2 u^2 v + 2\tau^2 u w_y, \\ -iv_t &= v_{xx} + \tau^2 v_{yy} + 2\tau^2 uv^2 + 2\tau^2 v w_y, \\ w_{xx} - \tau^2 w_{yy} &= 2\tau^2 (uv)_y.\end{aligned}\tag{3}$$

When $\tau = 1$, $v = \sigma \bar{u}^*$ ($\sigma = \pm 1$), (3) becomes the nonlocal Davey-Stewartson I equation (1). The Lax pair (3) becomes

$$\begin{aligned}\Phi_x &= U(\partial)\Phi \triangleq J\Phi_y + P\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_y + \begin{pmatrix} 0 & -u \\ \sigma \bar{u}^* & 0 \end{pmatrix} \Phi, \\ \Phi_t &= V(\partial)\Phi \triangleq -2iJ\Phi_{yy} - 2iP\Phi_y + iQ\Phi \\ &= -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{yy} - 2i \begin{pmatrix} 0 & -u \\ \sigma \bar{u}^* & 0 \end{pmatrix} \Phi_y \\ &\quad + i \begin{pmatrix} -\sigma u \bar{u}^* - w_y - w_x & u_x + u_y \\ \sigma(\bar{u}^*)_x - \sigma(\bar{u}^*)_y & \sigma u \bar{u}^* + w_y - w_x \end{pmatrix} \Phi\end{aligned}\tag{4}$$

where $\partial = \frac{\partial}{\partial y}$. Here $U(\partial)$ implies that U is a differential operator with respect to y .

The coefficients in the Lax pair (4) satisfies

$$\bar{J}^* = -KJK^{-1}, \quad \bar{P}^* = -KPK^{-1}, \quad \bar{Q}^* = -KQK^{-1}\tag{5}$$

where $K = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$. Here M^* refers to the complex conjugation (without transpose) of a matrix M . (5) implies

$$\overline{U(\partial)^*} = -KU(\partial)K^{-1}, \quad \overline{V(\partial)^*} = KV(\partial)K^{-1}.\tag{6}$$

Hence we have

Lemma 1 *If $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is a solution of (4), then so is $K\bar{\Phi}^* = \begin{pmatrix} \sigma \bar{\eta}^* \\ \bar{\xi}^* \end{pmatrix}$.*

By Lemma 1, take a solution $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ of (4) and let $H = \begin{pmatrix} \xi & \sigma \bar{\eta}^* \\ \eta & \bar{\xi}^* \end{pmatrix}$, then $G(\partial) = \partial - S$ with $S = H_y H^{-1}$ gives a Darboux transformation.[18, 19] Written explicitly,

$$S = \frac{1}{\xi \bar{\xi}^* - \sigma \eta \bar{\eta}^*} \begin{pmatrix} \bar{\xi}^* \xi_y - \sigma \eta (\bar{\eta}^*)_y & \sigma \xi (\bar{\eta}^*)_y - \sigma \bar{\eta}^* \xi_y \\ \bar{\xi}^* \eta_y - \eta (\bar{\xi}^*)_y & \xi (\bar{\xi}^*)_y - \sigma \bar{\eta}^* \eta_y \end{pmatrix}.\tag{7}$$

$G(\partial)$ also keeps the symmetries (5) invariant. After the action of $G(\partial)$, (u, w) is transformed to (\tilde{u}, \tilde{w}) by

$$G(\partial)U(\partial) + G_x(\partial) = \tilde{U}(\partial)G(\partial), \quad G(\partial)V(\partial) + G_t(\partial) = \tilde{V}(\partial)G(\partial).\tag{8}$$

That is,

$$\begin{aligned}\tilde{u} &= u + 2\sigma \frac{\bar{\eta}^* \xi_y - \xi(\bar{\eta}^*)_y}{\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta}, \\ \tilde{w} &= w + 2 \frac{(\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta)_y}{\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta}.\end{aligned}\tag{9}$$

The Darboux transformation of degree n is given by a matrix-valued differential operator

$$G(\partial) = \partial^n + G_1 \partial^{n-1} + \cdots + G_n\tag{10}$$

of degree n which is determined by

$$G(\partial)H_j = 0 \quad (j = 1, \dots, n)\tag{11}$$

for n matrix solutions H_j ($j = 1, \dots, n$) of (2). By comparing the coefficients of ∂^j in (8), the transformation of (P, Q) is

$$\tilde{P} = P - [J, G_1], \quad \tilde{Q} = Q + 2[J, G_2] - 2[JG_1 - P, G_1] + 4JG_{1,y} - 2nP_y.\tag{12}$$

Rewrite (11) as

$$\partial^n H_j + G_1 \partial^{n-1} H_j + \cdots + G_n H_j = 0 \quad (j = 1, \dots, n),\tag{13}$$

then

$$\begin{aligned}& \begin{pmatrix} G_1 & G_2 & \cdots & G_n \end{pmatrix} \begin{pmatrix} \partial^{n-1} H_1 & \partial^{n-1} H_2 & \cdots & \partial^{n-1} H_n \\ \partial^{n-2} H_1 & \partial^{n-2} H_2 & \cdots & \partial^{n-2} H_n \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & \cdots & H_n \end{pmatrix} \\ &= \begin{pmatrix} -\partial^n H_1 & -\partial^n H_2 & \cdots & -\partial^n H_n \end{pmatrix}.\end{aligned}\tag{14}$$

Write $H_j = \begin{pmatrix} h_{11}^{(j)} & h_{12}^{(j)} \\ h_{21}^{(j)} & h_{22}^{(j)} \end{pmatrix}$. By reordering the rows and columns, we have

$$\begin{pmatrix} (G_1)_{11} & \cdots & (G_n)_{11} & (G_1)_{12} & \cdots & (G_n)_{12} \\ (G_1)_{21} & \cdots & (G_n)_{21} & (G_1)_{22} & \cdots & (G_n)_{22} \end{pmatrix} W = -R\tag{15}$$

where $W = (W_{jk})_{1 \leq j, k \leq 2}$, $R = (R_{jk})_{1 \leq j, k \leq 2}$,

$$W_{jk} = \begin{pmatrix} \partial^{n-1} h_{jk}^{(1)} & \partial^{n-1} h_{jk}^{(2)} & \cdots & \partial^{n-1} h_{jk}^{(n)} \\ \partial^{n-2} h_{jk}^{(1)} & \partial^{n-2} h_{jk}^{(2)} & \cdots & \partial^{n-2} h_{jk}^{(n)} \\ \vdots & \vdots & & \vdots \\ h_{jk}^{(1)} & h_{jk}^{(2)} & \cdots & h_{jk}^{(n)} \end{pmatrix},\tag{16}$$

$$R_{jk} = \begin{pmatrix} \partial^n h_{jk}^{(1)} & \partial^n h_{jk}^{(2)} & \cdots & \partial^n h_{jk}^{(n)} \end{pmatrix} \quad (j, k = 1, 2). \quad (17)$$

Solving G from (15), we get the new solution of the equation (1) from (12). Especially,

$$\tilde{u} = u + 2(G_1)_{12}. \quad (18)$$

3 Soliton solutions

3.1 Single soliton solutions

Let $u = 0$, then $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ satisfies

$$\Phi_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_y, \quad \Phi_t = -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{yy}. \quad (19)$$

Take a special solution

$$\begin{aligned} \xi &= e^{\lambda x + \lambda y - 2i\lambda^2 t} + e^{-\lambda^* x - \lambda^* y - 2i\lambda^{*2} t}, \\ \eta &= ae^{\lambda x - \lambda y + 2i\lambda^2 t} + be^{-\lambda^* x + \lambda^* y + 2i\lambda^{*2} t}, \end{aligned} \quad (20)$$

where λ, a, b are complex constants. (9) gives the explicit solution

$$\tilde{u} = \frac{2\sigma\lambda_R(a^* - b^*)}{D} e^{2i\lambda_I y - 4i(\lambda_R^2 - \lambda_I^2)t} \quad (21)$$

of (1) where

$$\begin{aligned} D &= (2 - \sigma|a|^2 - \sigma|b|^2) \cosh(2\lambda_R y + 8\lambda_R \lambda_I t) \\ &\quad + \sigma(|a|^2 - |b|^2) \sinh(2\lambda_R y + 8\lambda_R \lambda_I t) \\ &\quad + 2(1 - \sigma \operatorname{Re}(ab^*)) \cosh(2\lambda_R x) - 2i\sigma \operatorname{Im}(ab^*) \sinh(2\lambda_R x). \end{aligned} \quad (22)$$

Here $z_R = \operatorname{Re} z$ and $z_I = \operatorname{Im} z$ for a complex number z . Since \tilde{w} is looked as an auxiliary function in (1), hereafter we will study mainly the behavior of \tilde{u} .

Note that \tilde{u} is global if $|a| < 1$ and $|b| < 1$ since $\operatorname{Re} D > 0$ in this case. Moreover, the peak moves in the velocity $(v_x, v_y) = (0, -4\lambda_I)$.

Remark 1 *The solution (21) may have singularities when the parameters are not chosen suitably, say, when $\sigma = -1$, $a = 1$, $b = -2$, or $\sigma = 1$, $a = 2$, $b = 1/4$.*

Figure 1 shows a 1 soliton solution with parameters $\sigma = -1$, $t = 20$, $\lambda = 0.07 - 1.5i$, $a = 0.2$, $b = 0.1i$.

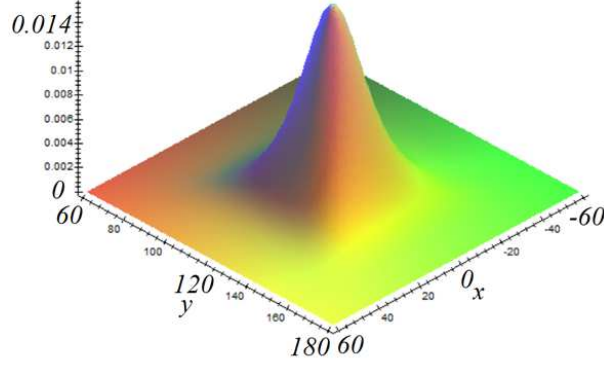


Figure 1: $|\tilde{u}|$ of a 1 soliton solution.

3.2 Multiple soliton solutions

For an $n \times n$ matrix M , define $\|M\| = \sup_{x \in \mathbf{C}^n, \|x\|=1} \|Mx\|$ where $\|\cdot\|$ is the standard Hermitian norm in \mathbf{C}^n . The following facts hold obviously.

- (i) $\|MN\| \leq \|M\| \|N\|$.
- (ii) Each entry M_{jk} of M satisfies $|M_{jk}| \leq \|M\|$.
- (iii) $|\det M| \leq \|M\|^n$.
- (iv) If $\|M\| < 1$, then $\|(I + M)^{-1}\| \leq (1 - \|M\|)^{-1}$.
- (v) If $\|M\| < 1$, then $|\det(I + M)| \geq (1 - \|M\|)^n$.

Now we construct explicit solutions according to (18). As in (20), take

$$\begin{aligned}\xi_k &= e^{\lambda_k(x+y)-2i\lambda_k^2 t} + e^{-\lambda_k^*(x+y)-2i\lambda_k^{*2} t}, \\ \eta_k &= a_k e^{\lambda_k(x-y)+2i\lambda_k^2 t} + b_k e^{-\lambda_k^*(x-y)+2i\lambda_k^{*2} t},\end{aligned}\tag{23}$$

then W_{jk} 's in (16) and R_{jk} 's in (17) are

$$\begin{aligned}(W_{11})_{jk} &= \lambda_k^{n-j} e_{k+} + (-\lambda_k^*)^{n-j} e_{k+}^{*-1}, & (W_{12})_{jk} &= \sigma a_k^* (-\lambda_k^*)^{n-j} e_{k+}^{*-1} + \sigma b_k^* \lambda_k^{n-j} e_{k+}, \\ (W_{21})_{jk} &= a_k (-\lambda_k)^{n-j} e_{k-} + b_k (\lambda_k^*)^{n-j} e_{k-}^{*-1}, & (W_{22})_{jk} &= (\lambda_k^*)^{n-j} e_{k-}^{*-1} + (-\lambda_k)^{n-j} e_{k-}, \\ (R_{11})_{1k} &= \lambda_k^n e_{k+} + (-\lambda_k^*)^n e_{k+}^{*-1}, & (R_{12})_{1k} &= \sigma a_k^* (-\lambda_k^*)^n e_{k+}^{*-1} + \sigma b_k^* \lambda_k^n e_{k+}, \\ (R_{21})_{1k} &= a_k (-\lambda_k)^n e_{k-} + b_k (\lambda_k^*)^n e_{k-}^{*-1}, & (R_{22})_{1k} &= (\lambda_k^*)^n e_{k-}^{*-1} + (-\lambda_k)^n e_{k-} \\ (j, k &= 1, \dots, n)\end{aligned}\tag{24}$$

where

$$e_{k\pm} = e^{\lambda_k(x \pm y) \mp 2i\lambda_k^2 t}.\tag{25}$$

However, temporarily, we assume e_{k+}, e_{k-} ($k = 1, \dots, n$) are arbitrary complex numbers rather than (25) holds.

Denote $L = \text{diag}((-1)^{n-1}, (-1)^{n-2}, \dots, -1, 1)$,

$$F = (\lambda_k^{n-j})_{1 \leq j, k \leq n}, \quad f = (\lambda_1^n, \dots, \lambda_n^n), \quad (26)$$

$$A = \text{diag}(a_1, \dots, a_n), \quad B = \text{diag}(b_1, \dots, b_n), \quad (27)$$

$$E_{\pm} = \text{diag}(e_{1\pm}, \dots, e_{n\pm}). \quad (28)$$

Then

$$W = \begin{pmatrix} FE_+ + LF^*E_+^{*-1} & \sigma FB^*E_+ + \sigma LF^*A^*E_+^{*-1} \\ LFAE_- + F^*BE_-^{*-1} & LFE_- + F^*E_-^{*-1} \end{pmatrix}, \quad (29)$$

$$R = \begin{pmatrix} fE_+ + (-1)^n f^*E_+^{*-1} & \sigma fB^*E_+ + \sigma(-1)^n f^*A^*E_+^{*-1} \\ (-1)^n fAE_- + f^*BE_-^{*-1} & (-1)^n fE_- + f^*E_-^{*-1} \end{pmatrix}. \quad (30)$$

By using the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{pmatrix} \quad (31)$$

for a block matrix where $\Delta = D - CA^{-1}B$, (15) gives

$$\left((G_1)_{12}, \dots, (G_n)_{12} \right) = -(RW^{-1})_{12} = -(R_{12} - R_{11}W_{11}^{-1}W_{12})\overset{\circ}{W}^{-1} \quad (32)$$

where

$$\begin{aligned} \overset{\circ}{W} &= W_{22} - W_{21}W_{11}^{-1}W_{12} \\ &= L \left(FE_- + LF^*E_-^{*-1} - \sigma(FAE_- + LF^*BE_-^{*-1}) \right. \\ &\quad \left. \cdot (FE_+ + LF^*E_+^{*-1})^{-1} (FB^*E_+ + LF^*A^*E_+^{*-1}) \right). \end{aligned} \quad (33)$$

Lemma 2 Suppose a_j and b_j are nonzero complex constants with $|a_j| < 1$, $|b_j| < 1$ ($j = 1, \dots, n$), $\kappa_1, \dots, \kappa_n$ are nonzero real constants with $|\kappa_j| \neq |\kappa_k|$ ($j, k = 1, \dots, n$; $j \neq k$), then there exist positive constants δ , C_1 and C_2 , which depend on a_j 's, b_j 's and κ_j 's, such that $|\det W| \geq C_1$ and

$$|(G_1)_{12}| \leq C_2 \max_{1 \leq k \leq n} \frac{|e_{k+}|}{1 + |e_{k+}|^2} \max_{1 \leq k \leq n} \frac{|e_{k-}|}{1 + |e_{k-}|^2} \quad (34)$$

hold whenever $|\lambda_j - i\kappa_j| < \delta$ and $e_{j\pm} \in \mathbf{C}$ ($j = 1, \dots, n$).

Proof. Denote $F^{-1}LF^* = I + Z$, then $Z = 0$ if $\lambda_1, \dots, \lambda_n$ are all purely imaginary. From (29) and (33),

$$\det W = \det(FE_+ + LF^*E_+^{*-1}) \det \overset{\circ}{W}, \quad (35)$$

$$\overset{\circ}{W} = L(FE_- + LF^*E_-^{*-1})(I - \sigma\chi_- \chi_+) \quad (36)$$

where

$$\begin{aligned} \chi_+ &= (FE_+ + LF^*E_+^{*-1})^{-1}(FB^*E_+ + LF^*A^*E_+^{*-1}) \\ &= \Xi_{1+}\Xi_{0+}^{-1} + \Xi_{0+}^{-1}(I + ZE_+^{*-1}\Xi_{0+}^{-1})^{-1}ZE_+^{*-1}(A^* - \Xi_{1+}\Xi_{0+}^{-1}), \\ \chi_- &= (FE_- + LF^*E_-^{*-1})^{-1}(FAE_- + LF^*BE_-^{*-1}) \\ &= \Xi_{1-}\Xi_{0-}^{-1} + \Xi_{0-}^{-1}(I + ZE_-^{*-1}\Xi_{0-}^{-1})^{-1}ZE_-^{*-1}(B - \Xi_{1-}\Xi_{0-}^{-1}), \end{aligned} \quad (37)$$

$$\Xi_{0\pm} = E_{\pm} + E_{\pm}^{*-1}, \quad \Xi_{1-} = AE_- + BE_-^{*-1}, \quad \Xi_{1+} = B^*E_+ + A^*E_+^{*-1}. \quad (38)$$

Let $c_0 = \max_{1 \leq k \leq n} \{|a_k|, |b_k|\} < 1$. Suppose $\|Z\| < \frac{1-c_0}{2}$, then we have the following estimates.

$$\|A\| \leq c_0 < 1, \quad \|B\| \leq c_0 < 1, \quad (39)$$

$$\|E_{\pm}\Xi_{0\pm}^{-1}\| \leq 1, \quad \|E_{\pm}^{*-1}\Xi_{0\pm}^{-1}\| \leq 1, \quad (40)$$

$$\|\Xi_{0\pm}\| \geq 2, \quad \|\Xi_{0\pm}^{-1}\| \leq \frac{1}{2}, \quad \|\Xi_{1\pm}\Xi_{0\pm}^{-1}\| \leq c_0 < 1, \quad (41)$$

$$\begin{aligned} \|E_+^{*-1}(\Xi_{1+}\Xi_{0+}^{-1} - A^*)\| &= \max_{1 \leq k \leq n} \frac{|a_k - b_k| |e_{k+}|}{1 + |e_{k+}|^2} \leq 1, \\ \|E_-^{*-1}(\Xi_{1-}\Xi_{0-}^{-1} - B)\| &= \max_{1 \leq k \leq n} \frac{|a_k - b_k| |e_{k-}|}{1 + |e_{k-}|^2} \leq 1, \end{aligned} \quad (42)$$

$$\|(I + ZE_{\pm}^{*-1}\Xi_{0\pm}^{-1})^{-1}\| \leq \|(1 - \|Z\| \|E_{\pm}^{*-1}\Xi_{0\pm}^{-1}\|)^{-1}\| \leq (1 - \|Z\|)^{-1} \leq 2. \quad (43)$$

Hence $\|\chi_{\pm} - \Xi_{1\pm}\Xi_{0\pm}^{-1}\| \leq \|Z\|$,

$$\|\chi_{\pm}\| \leq c_0 + \|Z\| \leq \frac{1+c_0}{2} < 1. \quad (44)$$

Denote

$$\pi_0 = |\det F| \Big|_{\substack{\lambda_j = i\kappa_j \\ j=1, \dots, n}}, \quad \pi_1 = \|F^{-1}\| \Big|_{\substack{\lambda_j = i\kappa_j \\ j=1, \dots, n}}, \quad \pi_2 = \|f\| \Big|_{\substack{\lambda_j = i\kappa_j \\ j=1, \dots, n}}. \quad (45)$$

Clearly, π_0, π_1, π_2 are all positive since $\det F|_{\substack{\lambda_j = i\kappa_j \\ j=1, \dots, n}}$ is a Vandermonde determinant. By the continuity, there exists $\delta > 0$ such that $\frac{\pi_0}{2} \leq |\det F| \leq 2\pi_0, \|F^{-1}\| \leq 2\pi_1, \|f\| \leq 2\pi_2$, and $\|F^{-1}LF^* - I\| = \|Z\| < \frac{1-c_0}{2}$ whenever $|\lambda_j - i\kappa_j| < \delta$. (35) and (36) lead to

$$\begin{aligned} |\det W| &= |\det F|^2 |\det \Xi_{0+}| |\det(I + ZE_+^{*-1}\Xi_{0+}^{-1})| \\ &\quad \cdot |\det \Xi_{0-}| |\det(I + ZE_-^{*-1}\Xi_{0-}^{-1})| |\det(1 - \sigma\chi_- \chi_+)| \\ &\geq \pi_0^2 (1 - \|Z\|)^{2n} (1 - \|\chi_+\| \|\chi_-\|)^n \geq \pi_0^2 \left(\frac{1+c_0}{2}\right)^{2n} \left(1 - \left(\frac{1+c_0}{2}\right)^2\right)^n, \end{aligned} \quad (46)$$

which is a uniform positive lower bound for any $e_{j\pm} \in \mathbf{C}$ ($j = 1, \dots, n$) when $|\lambda_j - i\kappa_j| < \delta$.

By (29), (30), (32) and (33),

$$\begin{aligned}
((G_1)_{12}, (G_2)_{12}, \dots, (G_n)_{12}) &= -(R_{12} - R_{11}W_{11}^{-1}W_{12})\overset{\circ}{W}^{-1} \\
&= -\sigma f E_+ \Xi_{0+}^{-1} (I + Z E_+^{*-1} \Xi_{0+}^{-1})^{-1} (I + Z)(B^* - A^*) E_+^{*-1} \overset{\circ}{W}^{-1} \\
&\quad - \sigma (-1)^n f^* E_+^{*-1} \Xi_{0+}^{-1} (I + Z E_+^{*-1} \Xi_{0+}^{-1})^{-1} (A^* - B^*) E_+ \overset{\circ}{W}^{-1} \\
&= -\sigma \left(f - (-1)^n f^* + (f E_+ \Xi_{0+}^{-1} + (-1)^n f^* E_+^{*-1} \Xi_{0+}^{-1}) (I + Z E_+^{*-1} \Xi_{0+}^{-1})^{-1} Z \right) \\
&\quad \cdot (B^* - A^*) E_+ E_+^{*-1} \Xi_{0+}^{-1} (I - \sigma \chi_- \chi_+)^{-1} \Xi_{0-}^{-1} (I + Z E_-^{*-1} \Xi_{0-}^{-1})^{-1} F^{-1} L^{-1}.
\end{aligned} \tag{47}$$

Here we have used $I + Z = (I + Z E_+^{*-1} \Xi_{0+}^{-1}) + Z E_+ \Xi_{0+}^{-1}$. Hence, by using (39)–(44),

$$\begin{aligned}
|(G_1)_{12}| &\leq 8(\|f\| + \|f^*\|) \|F^{-1}\| \|(I - \sigma \chi_- \chi_+)^{-1}\| \|\Xi_{0+}^{-1}\| \|\Xi_{0-}^{-1}\| \\
&\leq 64\pi_1\pi_2 \left(1 - \left(\frac{1+c_0}{2}\right)^2\right)^{-1} \max_{1 \leq k \leq n} \frac{|e_{k+}|}{1 + |e_{k+}|^2} \max_{1 \leq k \leq n} \frac{|e_{k-}|}{1 + |e_{k-}|^2}.
\end{aligned} \tag{48}$$

The lemma is proved.

Now we consider the solutions of the nonlocal Davey-Stewartson I equation. That is, we consider the case where $e_{j\pm}$'s are taken as (25).

Theorem 1 *Suppose a_j and b_j are nonzero complex constants with $|a_j| < 1$, $|b_j| < 1$ ($j = 1, \dots, n$), $\kappa_1, \dots, \kappa_n$ are nonzero real constants with $|\kappa_j| \neq |\kappa_k|$ ($j, k = 1, \dots, n$; $j \neq k$), then there exists a positive constant δ such that the following results hold for the derived solution $\tilde{u} = 2(G_1)_{12}$ of the nonlocal Davey-Stewartson I equation when $\text{Re } \lambda_j \neq 0$ and $|\lambda_j - i\kappa_j| < \delta$ ($j = 1, \dots, n$).*

(i) \tilde{u} is defined globally for $(x, y, t) \in \mathbf{R}^3$.

(ii) For fixed t , \tilde{u} tends to zero exponentially as $(x, y) \rightarrow 0$.

(iii) Let $y = \tilde{y} + vt$ and keep (x, \tilde{y}) bounded, then $\tilde{u} \rightarrow 0$ as $t \rightarrow \infty$ if $v \neq -4\lambda_{kI}$ for all k .

Proof. We have known that \tilde{u} is a solution of the nonlocal Davey-Stewartson equation in Section 2.

(i) According to Lemma 2, $|\det W|$ has a uniform positive lower bound. Hence \tilde{u} is defined globally.

(ii) When $x \geq 0$ and $y \geq 0$,

$$\begin{aligned}
|e_{k+}| &\geq e^{\lambda_{kR}\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} \text{ if } \lambda_{kR} > 0, \\
|e_{k+}| &\leq e^{-|\lambda_{kR}|\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} \text{ if } \lambda_{kR} < 0.
\end{aligned} \tag{49}$$

Hence $\max_{1 \leq k \leq n} \frac{|e_{k+}|}{1 + |e_{k+}|^2}$ tends to zero exponentially when $x \geq 0$, $y \geq 0$ and $(x, y) \rightarrow \infty$.

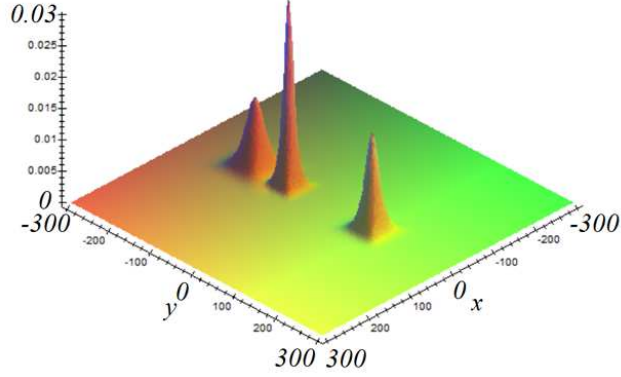


Figure 2: $|\tilde{u}|$ of a 3 soliton solution.

Likewise, we have

$$\begin{aligned}
|e_{k-}| &\leq e^{-\lambda_{kR}\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} \text{ if } x \leq 0, y \geq 0, \lambda_{kR} > 0, \\
|e_{k-}| &\geq e^{|\lambda_{kR}|\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} \text{ if } x \leq 0, y \geq 0, \lambda_{kR} < 0, \\
|e_{k+}| &\leq e^{-\lambda_{kR}\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} \text{ if } x \leq 0, y \leq 0, \lambda_{kR} > 0, \\
|e_{k+}| &\geq e^{|\lambda_{kR}|\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} \text{ if } x \leq 0, y \leq 0, \lambda_{kR} < 0, \\
|e_{k-}| &\geq e^{\lambda_{kR}\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} \text{ if } x \geq 0, y \leq 0, \lambda_{kR} > 0, \\
|e_{k-}| &\leq e^{-|\lambda_{kR}|\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} \text{ if } x \geq 0, y \leq 0, \lambda_{kR} < 0.
\end{aligned} \tag{50}$$

Lemma 2 implies that $\tilde{u} \rightarrow 0$ exponentially as $(x, y) \rightarrow \infty$.

(iii)

$$|e_{k\pm}| = e^{\lambda_{kR}(x \pm \tilde{y}) \pm \lambda_{kR}(v + 4\lambda_{kI})t}. \tag{51}$$

If $v \neq -4\lambda_{kI}$ for all $k = 1, \dots, n$, then either $e_{k+} \rightarrow 0$ or $e_{k+} \rightarrow \infty$ for all $k = 1, \dots, n$ when $t \rightarrow \infty$. Lemma 2 implies that $\tilde{u} \rightarrow 0$ when $t \rightarrow \infty$. The theorem is proved.

A 3 soliton solution is shown in Figure 2 where the parameters are $\sigma = -1$, $t = 20$, $\lambda_1 = 0.07 - 1.5i$, $\lambda_2 = 0.05 + 2i$, $\lambda_3 = 0.1 + i$, $a_1 = 0.2$, $a_2 = 0.1i$, $a_3 = 0.1$, $b_1 = 0.1i$, $b_2 = -0.2$, $b_3 = -0.2$. The figure of the solution appears similarly if σ is changed to $+1$, although it is not shown here.

4 “Line dark soliton” solutions

4.1 Single “line dark soliton” solutions

Now we take

$$u = \rho e^{-2i\sigma|\rho|^2 t}, \quad w = 0 \quad (52)$$

as a solution of (1) where ρ is a complex constant. The Lax pair (4) has a solution

$$\begin{pmatrix} e^{\alpha(\lambda)x + \beta(\lambda)y + \gamma(\lambda)t} \\ \frac{\lambda}{\rho} e^{\alpha(\lambda)x + \beta(\lambda)y + (\gamma(\lambda) + 2i\sigma|\rho|^2)t} \end{pmatrix}, \quad (53)$$

where

$$\begin{aligned} \alpha(\lambda) &= \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda} - \lambda \right), \quad \beta(\lambda) = \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda} + \lambda \right), \\ \gamma(\lambda) &= i(\alpha(\lambda)^2 - 2\alpha(\lambda)\beta(\lambda) - \beta(\lambda)^2) = i\lambda^2 - \frac{i}{2} \left(\frac{\sigma|\rho|^2}{\lambda} + \lambda \right)^2, \end{aligned} \quad (54)$$

λ is a complex constant. Note that $\alpha(-\lambda^*) = -(\alpha(\lambda))^*$, $\beta(-\lambda^*) = -(\beta(\lambda))^*$, $\gamma(-\lambda^*) = -(\gamma(\lambda))^*$.

Now take $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ where

$$\begin{aligned} \xi &= e^{\alpha x + \beta y + \gamma t} + e^{-\alpha^* x - \beta^* y - \gamma^* t}, \\ \eta &= \frac{\lambda}{\rho} e^{\alpha x + \beta y + (\gamma + 2i\sigma|\rho|^2)t} - \frac{\lambda^*}{\rho} e^{-\alpha^* x - \beta^* y - (\gamma^* - 2i\sigma|\rho|^2)t}. \end{aligned} \quad (55)$$

Here $\alpha = \alpha(\lambda)$, $\beta = \beta(\lambda)$, $\gamma = \gamma(\lambda)$. This Φ is a linear combination of the solutions of form (53). Then (9) gives the new solution

$$\tilde{u} = \rho e^{-2i\sigma|\rho|^2 t} \frac{\frac{\lambda^*}{\lambda} c_1 e^{2\beta_R y + 2\gamma_R t} + \frac{\lambda}{\lambda^*} c_1 e^{-2\beta_R y - 2\gamma_R t} - c_2 e^{2\alpha_R x} - c_2^* e^{-2\alpha_R x}}{c_1 (e^{2\beta_R y + 2\gamma_R t} + e^{-2\beta_R y - 2\gamma_R t}) + c_2 e^{2\alpha_R x} + c_2^* e^{-2\alpha_R x}} \quad (56)$$

of the nonlocal Davey-Stewartson I equation where

$$c_1 = 1 - \sigma \frac{|\lambda|^2}{|\rho|^2}, \quad c_2 = 1 + \sigma \frac{\lambda^2}{|\rho|^2} \quad (57)$$

This solution is smooth for all $(x, y, t) \in \mathbf{R}^3$ if $|\lambda| < |\rho|$.

Especially, if λ is real, then $\gamma_R = 0$, so we get a standing wave solution.

Figure 3 shows a 1 “line dark soliton” solution with parameters $\sigma = -1$, $t = 10$, $\rho = 1$, $\lambda = 0.3 + 0.1i$. The figure on the right describes the same solution but is upside down.

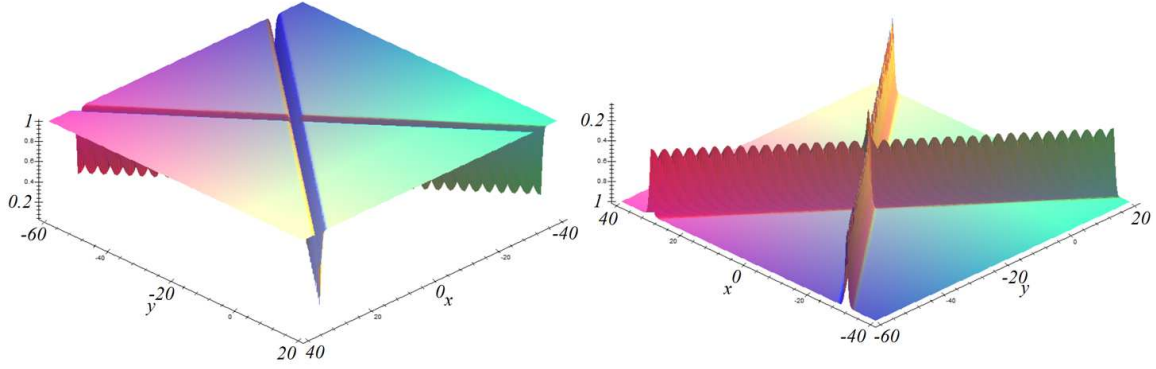


Figure 3: $|\tilde{u}|$ of a 1 “line dark soliton” solution.

4.2 Multiple “line dark soliton” solutions

Now we take n solutions

$$\begin{aligned}\xi_k &= e^{\alpha_k x + \beta_k y + \gamma_k t} + e^{-\alpha_k^* x - \beta_k^* y - \gamma_k^* t}, \\ \eta_k &= \frac{\lambda_k}{\rho} e^{\alpha_k x + \beta_k y + (\gamma_k + 2i\sigma|\rho|^2)t} - \frac{\lambda_k^*}{\rho} e^{-\alpha_k^* x - \beta_k^* y - (\gamma_k^* - 2i\sigma|\rho|^2)t}\end{aligned}\quad (58)$$

of form (55) to get multiple “line dark soliton” solutions where

$$\begin{aligned}\alpha_k &= \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda_k} - \lambda_k \right), \quad \beta_k = \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda_k} + \lambda_k \right), \\ \gamma_k &= i(\alpha_k^2 - 2\alpha_k\beta_k - \beta_k^2).\end{aligned}\quad (59)$$

Similar to (29) and (30), W and R in (15) are

$$W = \begin{pmatrix} FE_+ + LF^*E_+^{*-1} & -\sigma\rho^{*-1}e^{-i\phi}(LF\Lambda E_- - F^*\Lambda^*E_-^{*-1}) \\ \rho^{-1}e^{i\phi}(F\Lambda E_+ - LF^*\Lambda^*E_+^{*-1}) & LFE_- + F^*E_-^{*-1} \end{pmatrix}, \quad (60)$$

$$R = \begin{pmatrix} fE_+ + (-1)^n f^*E_+^{*-1} & -\sigma\rho^{*-1}e^{-i\phi}((-1)^n f\Lambda E_- - f^*\Lambda^*E_-^{*-1}) \\ \rho^{-1}e^{i\phi}(f\Lambda E_+ - (-1)^n f^*\Lambda^*E_+^{*-1}) & (-1)^n fE_- + f^*E_-^{*-1} \end{pmatrix}, \quad (61)$$

where

$$\begin{aligned}E_{\pm} &= \text{diag}(e_{k\pm})_{k=1,\dots,n}, \quad F = (\beta_k^{n-j})_{1 \leq j, k \leq n}, \quad f = (\beta_1^n, \dots, \beta_n^n), \\ \alpha_k &= \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda_k} - \lambda_k \right), \quad \beta_k = \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\lambda_k} + \lambda_k \right), \\ \gamma_k &= i(\alpha_k^2 - 2\alpha_k\beta_k - \beta_k^2),\end{aligned}\quad (62)$$

$\Lambda = \text{diag}(\lambda_k)_{1 \leq k \leq n}$, $\phi = 2\sigma|\rho|^2 t$. Moreover, $e_{k\pm} = e^{\alpha_k x \pm \beta_k y \pm \gamma_k t}$. However, as in the soliton case, we suppose temporarily that $e_{k\pm}$'s are arbitrary complex numbers.

Lemma 3 Suppose $\kappa_1, \dots, \kappa_n$ are distinct nonzero real numbers, then there exist positive constants ρ_0, δ, C_1 and C_2 , which depend on κ_j 's, such that $|\det W| \geq C_1$ and $|(G_1)_{12}| \leq C_2$ hold whenever $|\rho| > \rho_0$, $|\lambda_j - i\kappa_j| < \delta$ and $e_{j\pm} \in \mathbf{C}$ ($j = 1, \dots, n$).

Proof. Denote $F^{-1}LF^* = I + Z$, then $Z = 0$ if $\lambda_1, \dots, \lambda_n$ are all purely imaginary. Hence $\|Z\|$ is small enough if $|\lambda_1 - i\kappa_1|, \dots, |\lambda_n - i\kappa_n|$ are all small enough.

Let $c_0 = \max_{1 \leq k \leq n} |\kappa_k|$, $\pi_3 = |\det F| \Big|_{\substack{\lambda_k = i\kappa_k \\ k=1, \dots, n}}, \pi_4 = \|F^{-1}LF\| \Big|_{\substack{\lambda_k = i\kappa_k \\ k=1, \dots, n}}$. Then there exists δ with $0 < \delta < c_0$ such that $\|Z\| \leq \frac{1}{2}$, $|\det F| \geq \frac{\pi_3}{2}$, $\|F^{-1}LF\| \leq 2\pi_4$ if $|\lambda_k - i\kappa_k| < \delta$ ($k = 1, \dots, n$). In this case, $|\lambda_k| < c_0 + \delta < 2c_0$.

From (60),

$$\det W = \det(FE_+ + LF^*E_+^{*-1}) \det \overset{\circ}{W}, \quad (63)$$

where

$$\begin{aligned} \overset{\circ}{W} &= LFE_- + F^*E_-^{*-1} + \sigma|\rho|^{-2}(F\Lambda E_+ - LF^*\Lambda^*E_+^{*-1}) \\ &\quad \cdot (FE_+ + LF^*E_+^{*-1})^{-1}L(F\Lambda E_- - LF^*\Lambda^*E_-^{*-1}) \\ &= (1 + \sigma|\rho|^{-2}F\chi_+F^{-1}LF\chi_-F^{-1}L^{-1})LF(I + ZE_-^{*-1}\Xi_{0-}^{-1})\Xi_{0-} \end{aligned} \quad (64)$$

$$\begin{aligned} \chi_{\pm} &= F^{-1}(F\Lambda E_{\pm} - LF^*\Lambda^*E_{\pm}^{*-1})(FE_{\pm} + LF^*E_{\pm}^{*-1})^{-1}F \\ &= \Xi_{1\pm}\Xi_{0\pm}^{-1} - (Z\Lambda^* + \Xi_{1\pm}\Xi_{0\pm}^{-1}Z)E_{\pm}^{*-1}\Xi_{0\pm}^{-1}(I + ZE_{\pm}^{*-1}\Xi_{0\pm}^{-1})^{-1}, \end{aligned} \quad (65)$$

$$\Xi_{0\pm} = E_{\pm} + E_{\pm}^{*-1}, \quad \Xi_{1\pm} = \Lambda E_{\pm} - \Lambda^*E_{\pm}^{*-1}. \quad (66)$$

We have the following estimates:

$$\|E_{\pm}\Xi_{0\pm}^{-1}\| \leq 1, \quad \|E_{\pm}^{*-1}\Xi_{0\pm}^{-1}\| \leq 1, \quad \|\Xi_{1\pm}\Xi_{0\pm}^{-1}\| \leq 2c_0, \quad (67)$$

$$\|\Xi_{0\pm}\| \geq 2, \quad \|\Xi_{0\pm}^{-1}\| \leq \frac{1}{2}, \quad |\det \Xi_{0\pm}| \geq 2^n, \quad (68)$$

$$\|(I + ZE_{\pm}^{*-1}\Xi_{0\pm}^{-1})^{-1}\| \leq (1 - \|Z\|)^{-1} \leq 2, \quad |\det(I + ZE_{\pm}^{*-1}\Xi_{0\pm}^{-1})| \geq (1 - \|Z\|)^n \geq \frac{1}{2^n}. \quad (69)$$

Hence (65) implies

$$\|\chi_{\pm}\| \leq 2c_0 + 8c_0\|Z\| \leq 6c_0, \quad (70)$$

$$\|\chi_+F^{-1}LF\chi_-F^{-1}L^{-1}F\| \leq \|F^{-1}LF\|^2\|\chi_+\|\|\chi_-\| \leq 144c_0^2\pi_4^2. \quad (71)$$

By (63) and (64),

$$\begin{aligned} |\det W| &= |\det F|^2 |\det \Xi_{0+}| |\det \Xi_{0-}| |\det(I + ZE_+^{*-1}\Xi_{0+}^{-1})| \\ &\quad \cdot |\det(I + ZE_-^{*-1}\Xi_{0-}^{-1})| |\det(I + \sigma|\rho|^{-2}\chi_+F^{-1}LF\chi_-F^{-1}L^{-1}F)| \\ &\geq \frac{\pi_3^2}{4} (1 - 144c_0^2\pi_4^2|\rho|^{-2}) > 0 \end{aligned} \quad (72)$$

if $|\rho| > 12c_0\pi_4$. Therefore, $|\det W|$ has a uniform positive lower bound if $|\rho| > 12c_0\pi_4$ and $|\lambda_k - i\kappa_k| < \delta$ ($k = 1, \dots, n$).

By (15), (60), (61), (63) and (64),

$$\begin{aligned}
& ((G_1)_{12}, (G_2)_{12}, \dots, (G_n)_{12}) = -(R_{12} - R_{11}W_{11}^{-1}W_{12})\overset{\circ}{W}^{-1} \\
& = \sigma\rho^{*-1}e^{-i\phi}(-1)^n \left(f\Lambda E_- - (-1)^n f^*\Lambda^*E_-^{*-1} - (fE_+ + (-1)^n f^*E_+^{*-1}) \right. \\
& \quad \left. \cdot (-1)^n (FE_+ + LF^*E_+^{*-1})^{-1} L(F\Lambda E_- - LF^*\Lambda^*E_-^{*-1}) \right) \overset{\circ}{W}^{-1} \\
& = \sigma\rho^{*-1}e^{-i\phi}(-1)^n \left(f\Lambda E_- \Xi_{0-}^{-1} - (-1)^n f^*\Lambda^*E_-^{*-1}\Xi_{0-}^{-1} - \right. \\
& \quad \left. - (-1)^n (fE_+ \Xi_{0+}^{-1} + (-1)^n f^*E_+^{*-1}\Xi_{0+}^{-1})(I + ZE_+^{*-1}\Xi_{0+}^{-1})^{-1} F^{-1}LF \right. \\
& \quad \left. \cdot (\Xi_{1-}\Xi_{0-}^{-1} - Z\Lambda^*E_-^{*-1}\Xi_{0-}^{-1}) \right) (I + ZE_-^{*-1}\Xi_{0-}^{-1})^{-1} F^{-1}L^{-1} \cdot \\
& \quad \cdot (1 + \sigma|\rho|^{-2}F\chi_+F^{-1}LF\chi_-F^{-1}L^{-1})^{-1}.
\end{aligned} \tag{73}$$

$(G_1)_{12}$ is bounded when $|\rho| > 12c_0\pi_4$ and $|\lambda_k - i\kappa_k| < \delta$ ($k = 1, \dots, n$) because of the estimates (67)–(72). The lemma is proved.

Now we have the following theorem for the multiply “line dark soliton” solution.

Theorem 2 Suppose $\kappa_1, \dots, \kappa_n$ are distinct nonzero real numbers, then there exist positive constants ρ_0 and δ such that the following results hold for the derived solution $\tilde{u} = u + 2(G_1)_{12}$ of the nonlocal Davey-Stewartson I equation when $|\rho| > \rho_0$ and $|\lambda_j - i\kappa_j| < \delta$ ($j = 1, \dots, n$).

(i) \tilde{u} is globally defined and bounded for $(x, y, t) \in \mathbf{R}^3$.

(ii) Suppose the real numbers v_x, v_y satisfy $\alpha_{kR}v_x \pm \beta_{kR}v_y \neq 0$ for all $k = 1, \dots, n$ where α_k 's and β_k 's are given by (59), then $\lim_{s \rightarrow +\infty} |\tilde{u}| = |\rho|$ along the straight line $x = x_0 + v_x s$, $y = y_0 + v_y s$ for arbitrary $x_0, y_0 \in \mathbf{R}$.

Proof. (i) follows directly from Lemma 3. Now we prove (ii).

Since $|e_{k\pm}| = e^{(\alpha_{kR}v_x \pm \beta_{kR}v_y)s + (\alpha_{kR}x_0 \pm \beta_{kR}y_0 \pm \gamma_{kR}t)}$ along the straight line $x = x_0 + v_x s$, $y = y_0 + v_y s$, $\alpha_{kR}v_x \pm \beta_{kR}v_y \neq 0$ implies that for each k , $e_{k+} \rightarrow 0$ or $e_{k+} \rightarrow \infty$, and $e_{k-} \rightarrow 0$ or $e_{k-} \rightarrow \infty$ as $s \rightarrow +\infty$. Let

$$\begin{aligned}
\mu_k &= \begin{cases} \lambda_k & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y > 0, \\ -\lambda_k^* & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y < 0, \end{cases} \\
\nu_k &= \begin{cases} -\lambda_k & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y > 0, \\ \lambda_k^* & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y < 0, \end{cases} \\
a_k &= \begin{cases} \beta_k & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y > 0, \\ -\beta_k^* & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y < 0, \end{cases} \\
b_k &= \begin{cases} -\beta_k & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y > 0, \\ \beta_k^* & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y < 0, \end{cases}
\end{aligned} \tag{74}$$

then

$$a_k = \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\mu_k} + \mu_k \right), \quad b_k = \frac{1}{2} \left(\frac{\sigma|\rho|^2}{\nu_k} + \nu_k \right). \quad (75)$$

Rewrite (15) as

$$\begin{pmatrix} (G_1)_{11} & \cdots & (G_n)_{11} & \rho^{-1}e^{i\phi}(G_1)_{12} & \cdots & \rho^{-1}e^{i\phi}(G_n)_{12} \\ (G_1)_{21} & \cdots & (G_n)_{21} & \rho^{-1}e^{i\phi}(G_1)_{22} & \cdots & \rho^{-1}e^{i\phi}(G_n)_{22} \end{pmatrix} SW S^{-1} = -RS^{-1} \quad (76)$$

where $S = \begin{pmatrix} I_n & \\ & \rho e^{-i\phi} I_n \end{pmatrix}$, I_n is the $n \times n$ identity matrix.

Applying Cramer's rule to (76) and using (75), we have

$$\lim_{s \rightarrow +\infty} \tilde{u} = \lim_{s \rightarrow +\infty} (\rho e^{-i\phi} + 2(G_1)_{12}) = \rho e^{-i\phi} \left(1 - 2 \frac{\det W_1}{\det W_0} \right) = \rho e^{-i\phi} \frac{\det W_2}{\det W_0} \quad (77)$$

where

$$W_0 = \begin{pmatrix} a_1^{n-1} & \cdots & a_n^{n-1} & \sigma|\rho|^{-2}b_1^{n-1}\nu_1 & \cdots & \sigma|\rho|^{-2}b_n^{n-1}\nu_n \\ a_1^{n-2} & \cdots & a_n^{n-2} & \sigma|\rho|^{-2}b_1^{n-2}\nu_1 & \cdots & \sigma|\rho|^{-2}b_n^{n-2}\nu_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1 & \cdots & a_n & \sigma|\rho|^{-2}b_1\nu_1 & \cdots & \sigma|\rho|^{-2}b_n\nu_n \\ 1 & \cdots & 1 & \sigma|\rho|^{-2}\nu_1 & \cdots & \sigma|\rho|^{-2}\nu_n \\ a_1^{n-1}\mu_1 & \cdots & a_n^{n-1}\mu_n & b_1^{n-1} & \cdots & b_n^{n-1} \\ a_1^{n-2}\mu_1 & \cdots & a_n^{n-2}\mu_n & b_1^{n-2} & \cdots & b_n^{n-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1\mu_1 & \cdots & a_n\mu_n & b_1 & \cdots & b_n \\ \mu_1 & \cdots & \mu_n & 1 & \cdots & 1 \end{pmatrix}, \quad (78)$$

W_1 is obtained from W_0 by replacing the $(n+1)$ -th row with

$$\begin{pmatrix} a_1^n & \cdots & a_n^n & \sigma|\rho|^{-2}b_1^n\nu_1 & \cdots & \sigma|\rho|^{-2}b_n^n\nu_n \end{pmatrix}, \quad (79)$$

and

$$W_2 = \begin{pmatrix} a_1^{n-1} & \cdots & a_n^{n-1} & \sigma|\rho|^{-2}b_1^{n-1}\nu_1 & \cdots & \sigma|\rho|^{-2}b_n^{n-1}\nu_n \\ a_1^{n-2} & \cdots & a_n^{n-2} & \sigma|\rho|^{-2}b_1^{n-2}\nu_1 & \cdots & \sigma|\rho|^{-2}b_n^{n-2}\nu_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1 & \cdots & a_n & \sigma|\rho|^{-2}b_1\nu_1 & \cdots & \sigma|\rho|^{-2}b_n\nu_n \\ 1 & \cdots & 1 & \sigma|\rho|^{-2}\nu_1 & \cdots & \sigma|\rho|^{-2}\nu_n \\ -\sigma|\rho|^2a_1^{n-1}\mu_1^{-1} & \cdots & -\sigma|\rho|^2a_n^{n-1}\mu_n^{-1} & -\sigma|\rho|^{-2}b_1^{n-1}\nu_1^2 & \cdots & -\sigma|\rho|^{-2}b_n^{n-1}\nu_n^2 \\ a_1^{n-2}\mu_1 & \cdots & a_n^{n-2}\mu_n & b_1^{n-2} & \cdots & b_n^{n-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1\mu_1 & \cdots & a_n\mu_n & b_1 & \cdots & b_n \\ \mu_1 & \cdots & \mu_n & 1 & \cdots & 1 \end{pmatrix}. \quad (80)$$

Denote ROW_k and COL_k to be the k -th row and k -th column of W_2 respectively. The elementary transformations

$$\begin{aligned} \text{ROW}_{n+k+1} - 2 \cdot \text{ROW}_k &\rightarrow \text{ROW}_{n+k+1} \quad (k = 1, \dots, n-1), \\ \mu_k \cdot \text{COL}_k &\rightarrow \text{COL}_k \quad (k = 1, \dots, n), \\ \sigma|\rho|^2\nu_k^{-1} \cdot \text{COL}_{n+k} &\rightarrow \text{COL}_{n+k} \quad (k = 1, \dots, n), \\ -\sigma|\rho|^{-2} \cdot \text{ROW}_{n+k} &\rightarrow \text{ROW}_{n+k} \quad (k = 1, \dots, n), \\ \text{ROW}_k &\leftrightarrow \text{ROW}_{n+k} \quad (k = 1, \dots, n) \end{aligned} \quad (81)$$

transform W_2 to W_0 . Hence $\det W_2 = \prod_{k=1}^n \frac{\nu_k}{\mu_k} \det W_0$. This leads to $\lim_{s \rightarrow +\infty} |\tilde{u}| = |\rho|$ since

$$\prod_{k=1}^n |\mu_k| = \prod_{k=1}^n |\nu_k| = \prod_{k=1}^n |\lambda_k|. \quad \text{The theorem is proved.}$$

A 2 “line dark soliton” solution is shown in Figure 4 where the parameters are $\sigma = -1$, $t = 10$, $\rho = 1$, $\lambda_1 = 0.8 + 0.1i$, $\lambda_2 = -0.6 - 0.3i$. The figure on the right describes the same solution but is upside down.

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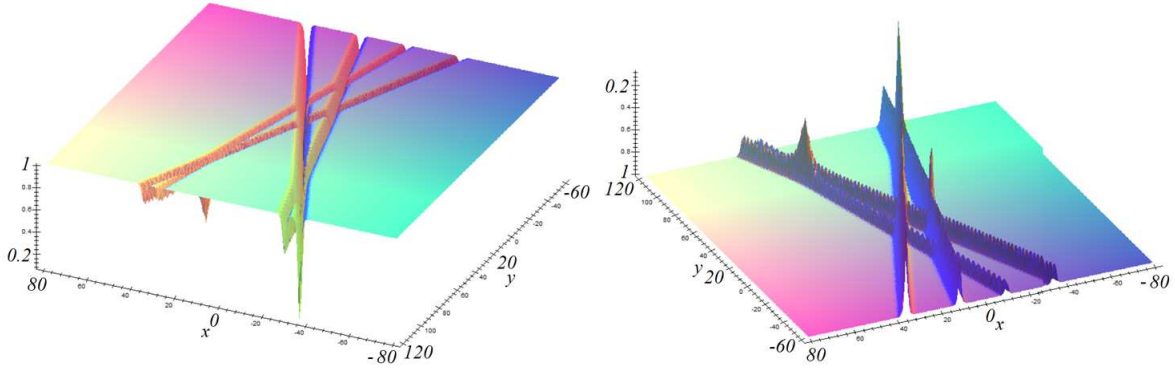


Figure 4: $|\tilde{u}|$ of a 2 “line dark soliton” solution.

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